

# G-ACTIONS ON GRAPHS

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ABSTRACT. Let  $G$  be an  $n$ -dimensional torus and  $\tau$  a Hamiltonian action of  $G$  on a compact symplectic manifold,  $M$ . If  $M$  is pre-quantizable one can associate with  $\tau$  a representation of  $G$  on a virtual vector space,  $Q(M)$ , by  $\text{spin}^{\mathbb{C}}$ -quantization. If  $M$  is a symplectic GKM manifold we will show that several well-known theorems about this “quantum action” of  $G$ : for example, the convexity theorem, the Kostant multiplicity theorem and the “quantization commutes with reduction” theorem for circle subgroups of  $G$ , are basically just theorems about  $G$ -actions on graphs.

## 1. INTRODUCTION

Let  $\Gamma$  be a finite  $d$ -valent graph and let  $G$  be a  $n$ -dimensional torus. In this paper we will be concerned with objects (rings, modules,  $G$ -representations ...) associated to an “action” of  $G$  on  $\Gamma$ . To define what we mean by this term, let  $V_{\Gamma} = V$  be the vertices of  $\Gamma$  and  $E_{\Gamma}$  the oriented edges. For each  $e \in E_{\Gamma}$  let  $i(e)$  and  $t(e)$  be the initial and terminal vertices of  $e$ , and let  $\bar{e}$  be the edge,  $e$ , with its orientation reversed. (Thus  $i(e) = t(\bar{e})$  and  $t(e) = i(\bar{e})$ .)

**Definition 1.1.** Let  $\varrho$  be a map which assigns to each oriented edge,  $e$ , of  $\Gamma$  a one dimensional representation,  $\varrho_e$ , with character

$$\chi_e : G \rightarrow S^1, \quad (1.1)$$

let  $\tau$  be a map which assigns to each vertex,  $p$ , of  $\Gamma$  a  $d$ -dimensional representation,  $\tau_p$ , and let  $G_e$  be the kernel of (1.1).  $\varrho$  and  $\tau$  define an *action* of  $G$  on  $\Gamma$  if they satisfy the axioms (1.2)–(1.4) below :

$$\tau_p \simeq \bigoplus_{i(e)=p} \varrho_e \quad (1.2)$$

$$\varrho_{\bar{e}} \simeq \varrho_e^* \quad (1.3)$$

$$\tau_{i(e)}|_{G_e} \simeq \tau_{t(e)}|_{G_e}. \quad (1.4)$$

**Remark.** For the connection between this graph-theoretic notion of “ $G$ -action” and the usual notion of  $G$ -action, see Section 7 (or, for more details, [GZ2, § 3.1]).

Let  $\mathbb{Z}_G^*$  be the weight lattice of  $G$ , and let  $\alpha_e \in \mathbb{Z}_G^*$  be the weight of the representation,  $\varrho_e$ , *i.e.*

$$\chi_e = e^{2\pi i \alpha_e}. \quad (1.5)$$

By (1.2) and (1.5) both  $\varrho$  and  $\tau$  are determined by the  $\alpha_e$ ’s; so an action of  $G$  on a graph,  $\Gamma$ , can be thought of as a labeling of each edge,  $e$ , of the graph by a weight,

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$\alpha_e$ . This labeling, however, will be forced by (1.2)–(1.4) to satisfy certain axioms. For instance, by (1.3)

$$\alpha_{\bar{e}} = -\alpha_e. \quad (1.6)$$

We will say that an action is a *GKM action* if, for every pair of edges with the same initial vertex,  $p = i(e_1) = i(e_2)$ , either  $e_1 = e_2$  or  $\alpha_{e_1}$  and  $\alpha_{e_2}$  are linearly independent. (For the geometric interpretation of this property, see Section 7). All the actions we consider below will be assumed to be GKM actions.

This paper is the fourth in a series of papers on the equivariant cohomology of graphs. The first three papers in this series were concerned with the equivariant cohomology ring,  $H_G(\Gamma)$ . In this paper we will be concerned with a slightly more complicated object: the equivariant “ $K$ -cohomology” ring of  $\Gamma$ . However, to motivate its definition, we will first recall how  $H_G(\Gamma)$  is defined: Denote by  $\mathfrak{g}$  and  $\mathfrak{g}_e$  the Lie algebras of  $G$  and  $G_e$ , and let  $\mathbb{S}(\mathfrak{g}^*)$  and  $\mathbb{S}(\mathfrak{g}_e^*)$  be the symmetric algebras over the duals of  $\mathfrak{g}$  and  $\mathfrak{g}_e$ . From the inclusion of  $\mathfrak{g}_e$  into  $\mathfrak{g}$  one gets a restriction map

$$r_e : \mathbb{S}(\mathfrak{g}^*) \rightarrow \mathbb{S}(\mathfrak{g}_e^*). \quad (1.7)$$

**Definition 1.2.**  $H_G(\Gamma)$  is the set of all functions,  $f : V_\Gamma \rightarrow \mathbb{S}(\mathfrak{g}^*)$ , which satisfy the compatibility conditions

$$r_e f_{i(e)} = r_e f_{t(e)} \quad (1.8)$$

for all edges,  $e$  of  $\Gamma$ .

Following [KR] we will define the  $K$ -theory analog of  $H_G(\Gamma)$  simply by replacing  $\mathbb{S}(\mathfrak{g}^*)$  in the definition by the *representation ring*,  $R(G)$ , of  $G$ .

**Definition 1.3.**  $K_G(\Gamma)$  is the set of all functions,  $f : V_\Gamma \rightarrow R(G)$ , which satisfy the compatibility condition (1.8),  $r_e$  being the restriction map,  $R(G) \rightarrow R(G_e)$ .

**Remarks.** 1. Since  $G$  is an  $n$ -torus, the representation ring  $R(G)$  can be identified with the *character ring* of  $G$ , i.e. the ring of all finite sums

$$\sum m_k e^{2\pi i \alpha_k} \quad (1.9)$$

with  $m_k \in \mathbb{Z}$  and  $\alpha_k \in \mathbb{Z}_G^*$ . We will frequently use this identification, referring to a representation by indicating the element of the character ring it corresponds to and vice-versa.

2. Point-wise multiplication makes  $K_G(\Gamma)$  into a ring. Moreover, since the *constant* functions satisfy (1.8), this ring contains the ring,  $R(G)$ , as a subring.

Given  $f \in K_G(\Gamma)$  let

$$\chi(f) = \sum_{p \in V} f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1}. \quad (1.10)$$

We will call  $\chi(f)$  the *Atiyah-Bott character* of the class,  $f$ . The individual summands on the right hand side are elements of a quotient ring of  $R(G)$ ; however, we will prove

**Theorem 1.1.** *The sum (1.10) is an element of  $R(G)$ .*

Thus (1.10) defines a morphism of  $R(G)$ -modules

$$\chi : K_G(\Gamma) \rightarrow R(G)$$

which we will call the *character map*. A helpful way of looking at this map is in terms of virtual representations. Namely, to each  $p \in V$ , one can attach an infinite-dimensional virtual representation,  $Q(\tau_p)$ , the “ $\text{spin}^{\mathbb{C}}$ -quantization” of the action,  $\tau_p$ , of  $G$  on  $\mathbb{C}^d$ , and Theorem 1.1 asserts that the sum

$$Q(f) = \bigoplus_{p \in V} Q(\tau_p) \otimes f_p \quad (1.11)$$

is a finite dimensional virtual representation and that its character is given by (1.10).

Suppose, in particular, that  $f$  has the form

$$f_p = e^{2\pi i \alpha_p}, \quad \alpha_p \in \mathbb{Z}_G^*. \quad (1.12)$$

Then by (1.8)

$$\alpha_q - \alpha_p = m_e \alpha_e \quad (1.13)$$

for every pair of vertices,  $p$  and  $q$ , and edge,  $e$ , joining  $p$  to  $q$ .

**Definition 1.4.**  $f$  is symplectic if  $m_e > 0$  for all  $e$ .

If  $f$  is symplectic, the representation (1.11) has the following convexity property. (Compare with [GS, Theorem 6.3].)

**Theorem 1.2.** *If  $\alpha$  is a weight of  $Q(f)$  then  $\alpha$  is in the convex hull of  $\{\alpha_p; p \in V\}$ .*

Let's denote this convex hull by  $\Delta$ . We will call a weight,  $\alpha$ , an *extremal* weight if it is a vertex of  $\Delta$ . For these weights we will prove

**Theorem 1.3.** *If  $\alpha$  is extremal, it occurs in  $Q(f)$  with multiplicity 1.*

For non-extremal weights we will prove a more refined result. Fix a vector,  $\xi$ , in  $\mathfrak{g}$  with the property  $\alpha_e(\xi) \neq 0$  for all edges,  $e$ , of  $\Gamma$ ; given a vertex,  $p$ , let

$$\mathcal{E}_p = \{e \in E_\Gamma; p \text{ is a vertex of } e \text{ and } \alpha_e(\xi) > 0\},$$

and let  $\sigma_p$  be the number of edges  $e \in \mathcal{E}_p$  for which  $p = t(e)$ .

For  $e \in \mathcal{E}_p$  define

$$(-1)^e = \begin{cases} 1 & \text{if } p = i(e) \\ -1 & \text{if } p = t(e) \end{cases},$$

and let

$$\begin{aligned} (-1)^p &= \prod_{e \in \mathcal{E}_p} (-1)^e = (-1)^{\sigma_p} \\ \delta_p &= \frac{1}{2} \sum_{e \in \mathcal{E}_p} \alpha_e, \\ \delta_p^\# &= \frac{1}{2} \sum_{e \in \mathcal{E}_p} (-1)^e \alpha_e. \end{aligned}$$

**Definition 1.5.** The Kostant partition function

$$N_p : \mathbb{Z}_G^* \rightarrow \mathbb{N}$$

is the function which assigns to every weight,  $\alpha$ , the number of distinct ways in which  $\alpha$  can be written as a sum

$$\alpha = \sum n_e \alpha_e, \quad e \in \mathcal{E}_p$$

with non-negative integer coefficients.

**Theorem 1.4.** *The multiplicity with which a weight,  $\alpha$ , occurs in  $Q(f)$  is equal to*

$$\sum_p (-1)^p N_p(\alpha - \alpha_p + \delta_p^\# - \delta_p). \quad (1.14)$$

(Compare with [GLS, (1.13)].)

The next results which we will describe involve a graph-theoretical analog of the notion of “reduction by a circle action” in symplectic geometry. Let  $T$  be a circle subgroup of  $G$  which is not contained in any of the groups,  $G_e$ . Then if  $\xi$  is the infinitesimal generator of  $T$

$$\alpha_e(\xi) \neq 0 \quad (1.15)$$

for all  $e$ . A function,  $\phi : V \rightarrow \mathbb{R}$  is called a  $T$ -moment map if for all edges  $e \in E_\Gamma$

$$\frac{\phi(t(e)) - \phi(i(e))}{\alpha_e(\xi)} > 0. \quad (1.16)$$

We recall ([GZ1, § 2.2]) that there is a simple necessary and sufficient condition for the existence of such a map. By (1.6) one can orient  $\Gamma$  by assigning to each unoriented edge the orientation for which  $\alpha_e(\xi) > 0$ . Then, for the existence of a  $T$ -moment map, it is necessary and sufficient that this graph *have no oriented cycles*. We will call the numbers,  $\phi(p)$ , the *critical values* of  $\phi$ . By perturbing  $\phi$  slightly one can arrange that these  $\phi(p)$ ’s are all distinct.

Let  $c \in \mathbb{R}$  be a regular (non-critical) value of  $\phi$ , and let  $V_c$  be the set of all oriented edges,  $e$ , of  $\Gamma$  with  $\phi(t(e)) > c > \phi(i(e))$ . One can make  $V_c$  into the set of vertices of a new object,  $\Gamma_c$ , and this object is our graph-theoretical “reduction of  $\Gamma$  at  $c$ ”. (Unfortunately,  $\Gamma_c$  is not a graph. It is a slightly more complicated object: a “hypergraph”. For details see [GZ3, § 3].)

Now fix an element,  $f$ , of  $K_G(\Gamma)$ , and for every edge,  $e$ , in  $V_c$  let  $p = i(e)$  and let

$$\hat{f}_e = f_p \prod_{e'} (1 - e^{2\pi i \alpha_{e'}})^{-1} \quad (1.17)$$

the product being over all  $e'$  with  $i(e') = p$  and  $e' \neq e$ . By composing the inclusion map of  $G_c$  into  $G$  with the projection of  $G$  onto  $G/T$ , one gets a surjective finite-to-one map  $\pi_e : G_e \rightarrow G/T$  and hence a “push-forward” in  $K$ -theory (see Section 3)

$$(\pi_e)_* : R(G_e) \rightarrow R(G/T).$$

This can be formally extended to elements of the quotient ring of  $R(G_e)$  of the form (1.17), and by applying it to (1.17) one gets, for every vertex of  $\Gamma_c$ , an element

$$f_c^\#(e) = (\pi_e)_* r_e \hat{f}_e \quad (1.18)$$

of a quotient ring of  $R(G/T)$ .

**Theorem 1.5.** *The sum*

$$\chi_c(f) = \sum_{e \in V_c} f_c^\#(e)$$

*is in  $R(G/T)$ .*

We will prove this by proving a stronger result. Let

$$f_p^\# = f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1}$$

be the  $p^{\text{th}}$  summand on the right hand side of (1.10); and, for  $g \in G$ , consider the integral over  $T$

$$\int f_p^\#(gt) dt. \quad (1.19)$$

We will see in Section 5 that the integrand has poles at a finite number of points,  $t_i \in T$  so this integral as it stands isn't well defined. However, one can “regularize” it by moving the contour of integration to a curve in  $T^\mathbb{C}$  which surrounds the  $t_i$ 's; and, denoting this regularized integral by  $\text{Res}_T f_p^\#$  we will prove:

**Theorem 1.6.**  *$\text{Res}_T f_p^\#$  is an element of  $R(G/T)$ .*

Our strengthened version of Theorem 1.5 asserts:

**Theorem 1.7.**  *$\chi_c(f)$  is equal to the sum*

$$\sum_{\phi(p) > c} \text{Res}_T f_p^\#. \quad (1.20)$$

Next we will explain what “quantization commutes with reduction” translates into the context of graphs. Recall that an element,  $f$ , of  $K_G(\Gamma)$  of the form

$$f_p = e^{2\pi i \alpha_p}, \quad \alpha_p \in \mathbb{Z}_G^*$$

is *symplectic* if

$$\alpha_q - \alpha_p = m_e \alpha_e, \quad m_e > 0$$

for every pair of vertices,  $p$  and  $q$ , and edge,  $e$  joining  $p$  to  $q$ . For  $f$  symplectic, the map

$$\phi : V \rightarrow \mathbb{R}, \quad p \rightarrow \alpha_p(\xi)$$

is a  $T$ -moment map. Assume zero is a regular value of this map, i.e.  $\alpha_p(\xi) \neq 0$  for all  $p$ ; and let  $\Gamma_{\text{red}} = \Gamma_0$  and  $\chi_{\text{red}} = \chi_0$ .

**Theorem 1.8.** *Let  $Q(\Gamma)$  be the virtual representation of  $G$  with character,  $\chi(f)$  and  $Q(\Gamma_{\text{red}})$  the virtual representation of  $G/T$  with character,  $\chi_{\text{red}}(f)$ . Then, as virtual representations of  $G/T$*

$$Q(\Gamma_{\text{red}}) = Q(\Gamma)^T. \quad (1.21)$$

Finally in the last section of this paper we will show that if  $M$  is a GKM manifold and  $\Gamma$  is its “one-skeleton”, these theorems about graphs have  $K$ -theoretic implications for  $M$  (thanks to a beautiful recent result of Allen Knutson and Ioanid Rosu which asserts that  $K_G(M) \otimes \mathbb{C} \simeq K_G(\Gamma) \otimes \mathbb{C}$ ).

## 2. SOME ALGEBRAIC PRELIMINARIES

We will collect in this section some elementary facts about lattices and tori which will be needed in the proofs. Let  $V$  be an  $n$ -dimensional real vector space and let  $L$  be a rank  $n$  lattice sitting inside  $V$ . Let

$$L^* = \{\alpha \in V^*; \alpha(v) \in \mathbb{Z} \text{ for all } v \in L\}$$

be the dual lattice in  $V^*$ . An element of  $L$  is *primitive* if it is not of the form,  $kv$ , with  $v \in L$  and  $|k| > 1$ .

**Lemma 2.1.**  *$v \in L$  is primitive if and only if there is an  $\alpha \in L^*$  with  $\alpha(v) = 1$ .*

**Lemma 2.2.**  *$v$  is primitive if and only if there exists a basis  $v_1, \dots, v_n$  of  $L$  with  $v = v_1$ .*

Now let  $G$  be an  $n$ -dimensional torus and let  $\mathfrak{g}$  be its Lie algebra.

**Definition 2.1.** The *group lattice* of  $G$ ,  $\mathbb{Z}_G$ , is the kernel of the exponential map,  $\exp : \mathfrak{g} \rightarrow G$  and its dual,  $\mathbb{Z}_G^*$ , is the *weight lattice* of  $G$ .

In particular

$$G = \mathfrak{g}/\mathbb{Z}_G$$

and the exponential map is just the projection of  $\mathfrak{g}$  onto  $\mathfrak{g}/\mathbb{Z}_G$ . Given a weight,  $\alpha \in \mathbb{Z}_G^*$ , let  $\chi_\alpha$  be the character of  $G$  defined by

$$\chi_\alpha(g) = e^{2\pi i \alpha(x)}, \quad g = \exp(x).$$

**Proposition 2.1.** *If  $\alpha$  is primitive, the subgroup*

$$G_\alpha = \{g \in G; \chi_\alpha(g) = 1\} \tag{2.1}$$

*is connected, i.e. is an  $(n-1)$ -dimensional subtorus of  $G$ . More generally, if  $\beta$  is primitive and  $\alpha = k\beta$ ,  $k > 1$ , the identity component of  $G_\alpha$  is  $G_\beta$  and  $G_\alpha/G_\beta$  is a finite cyclic group of order  $k$ .*

Let  $\xi$  be a primitive element of  $\mathbb{Z}_G$  and let

$$G_\xi = \{\exp(t\xi); 0 \leq t < 1\}. \tag{2.2}$$

then  $G_\xi$  is a closed, connected one-dimensional subgroup of  $G$ .

**Proposition 2.2.** *If  $\alpha(\xi) = 0$ , then  $G_\xi \subset G_\alpha$  and if  $\alpha(\xi) \neq 0$ , then  $G_\xi \cap G_\alpha$  is a finite cyclic subgroup of order  $|\alpha(\xi)|$ .*

Let  $G_1 = G/G_\xi$  and let  $\gamma : G_\alpha \rightarrow G_1$  be the composition of the inclusion,  $G_\alpha \rightarrow G$ , and the projection,  $G \rightarrow G_1$ .

**Corollary 2.1.** *The map  $\gamma$  is surjective and its kernel is a cyclic subgroup of  $G_\alpha$  of order  $|\alpha(\xi)|$ .*

3. THE REPRESENTATION RING  $R(G)$ 

The groups,  $G$ , in this section will be compact commutative Lie groups. For such a group every irreducible representation is one-dimensional, *i.e.* is defined by a homomorphism of  $G$  into  $S^1$ . Thus the elements of  $R(G)$  can be identified with the *character ring* of  $G$ : all finite sums of the form

$$\sum m_i \chi_i \quad (3.1)$$

$m_i$  being an integer and  $\chi_i$  a homomorphism of  $G$  into  $S^1$  (or “character”.) Hence, if  $G$  is an  $n$ -torus, (3.1) is a sum of the form (1.9).

In this section we will discuss some functorial properties of this ring. First we note that  $R(G)$  is naturally a contravariant functor, *i.e.* if  $\gamma : G \rightarrow H$  is a homomorphism of Lie groups, then a representation of  $H$  can be converted, by composition with  $\gamma$ , into a representation of  $G$ ; so there is a natural map

$$\gamma^* : R(H) \rightarrow R(G) \quad (3.2)$$

and it is easy to see that this is an algebra homomorphism. A much more interesting object for us will be a map in the opposite direction

$$\gamma_* : R(G) \rightarrow R(H) \quad (3.3)$$

which we will define here modulo the assumption

$$(*) \quad \text{the kernel and cokernel of } \gamma \text{ are finite.}$$

First let's assume that  $\gamma$  is surjective, *i.e.* that  $H = G/W$  and that  $W$  is a finite subgroup of  $G$ . Let  $\rho$  be a representation of  $G$  on a vector space,  $V$ , and let  $V^W$  be the vectors in  $V$  which transform trivially under  $W$ . Then the restriction of  $\rho$  to  $V^W$  is a representation,  $\rho^W$ , of  $G/W$  and  $\gamma_*$  is the map defined by  $\rho \rightarrow \rho^W$ .

Next assume that  $\gamma$  is injective, *i.e.* that  $G$  is a closed subgroup of  $H$  and  $G \setminus H$  is finite. Given a representation,  $\rho$ , of  $G$  on a vector space,  $V$ , let  $\rho_{ind}$  be the induced representation of  $H$  (*i.e.* let  $V_{ind}$  be the vector space consisting of maps  $f : H \rightarrow V$  which satisfy  $f(gh) = \rho(g)f(h)$  and let

$$(\rho_{ind}f)(k) = f(kh^{-1})$$

for all  $k \in H$ ). In this case  $\gamma_*$  is the map defined by  $\rho \rightarrow \rho_{ind}$ .

Finally if  $\gamma$  is neither injective nor surjective, let  $G_1$  be the image of  $\gamma$ . Then  $\gamma$  factors into the submersion  $\gamma_1 : G \rightarrow G_1$ , composed with the inclusion,  $\gamma_2 : G_1 \rightarrow H$ , and one defines

$$\gamma_* = (\gamma_2)_*(\gamma_1)_*. \quad (3.4)$$

This map is unfortunately not a ring homomorphism, but it is a morphism of  $R(H)$ -modules: for  $\chi \in R(G)$  and  $\tau \in R(H)$

$$\gamma_*(\chi\gamma^*\tau) = (\gamma_*\chi)\tau. \quad (3.5)$$

We will mostly be interested in the case when  $\gamma$  is a submersion, *i.e.* when  $H = G/W$ . In this case one has an alternative way of looking at  $\gamma$ :

**Lemma 3.1.** *Let  $\rho$  be a unitary representation of  $G$  on a complex vector space,  $V$ . Then the orthogonal projection of  $V$  onto  $V^W$  is given by the operator*

$$P = \frac{1}{|W|} \sum_{w \in W} \rho(w). \quad (3.6)$$

*Proof.* If  $v \in V^W$ ,  $\rho(w)v = v$ , so  $Pv = v$ . Moreover, for all  $v \in V$  and  $a \in W$ ,

$$\rho(a)Pv = \frac{1}{|W|} \sum_{w \in W} \rho(aw)v = Pv,$$

so  $Pv \in V^W$ . Finally,

$$P^* = \frac{1}{|W|} \sum_{w \in W} \rho(w^{-1}) = P,$$

so  $P$  is the *orthogonal* projection of  $V$  onto  $V^W$ .  $\square$

**Corollary 3.1.** *Let  $g$  be an element of  $G$  and let  $\bar{g}$  be its image in  $G/W$ . Then*

$$(\gamma_*\rho)(\bar{g}) = \frac{1}{|W|} \sum_{w \in W} \rho(gw). \quad (3.7)$$

*In particular, let  $f : G \rightarrow \mathbb{C}$  be the function (3.1), i.e.*

$$f(g) = \sum m_i \chi_i(g). \quad (3.8)$$

*Then*

$$\gamma_*f(\bar{g}) = \frac{1}{|W|} \sum_{w \in W} f(gw). \quad (3.9)$$

#### 4. CONVEXITY AND MULTIPLICITIES

*The proof of Theorem 1.1. :*

Let  $\alpha_1, \dots, \alpha_N$  be primitive vectors such that for every  $e \in E_\Gamma$  there exists a unique  $k \in \{1, \dots, N\}$  such that  $\alpha_e$  is a multiple of  $\alpha_k$ . If  $m_1\alpha_1, \dots, m_s\alpha_1$  are all the occurrences of multiples of  $\alpha_1$  among all the weights, let  $M_1 = \text{l.c.m.}(m_1, \dots, m_s)$ . Similarly we define  $M_2, \dots, M_N$ . Then

$$\chi(f) = \frac{g}{\prod_{j=1}^N (1 - e^{2\pi i M_j \alpha_j})} \quad (4.1)$$

with  $g \in R(G)$ . We will show that  $1 - e^{2\pi i M_1 \alpha_1}$  divides  $g$  in  $R(G)$ .

The vertices of  $\Gamma$  can be divided into two categories:

1. The first subset,  $V_1$ , contains the vertices  $p \in V_\Gamma$  for which none of the  $\alpha_e$ 's with  $i(e) = p$ , is a multiple of  $\alpha_1$
2. The second subset,  $V_2$ , contains the vertices  $p \in V_\Gamma$  for which there exists an edge  $e$  such that  $i(e) = p$  and  $\alpha_e$  is a multiple of  $\alpha_1$ . (Notice that there will be exactly one such edge.)

The part of (1.10) corresponding to vertices in the first category will then be of the form

$$\sum_{p \in V_1} f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} = g_1 \prod_{j=2}^N (1 - e^{2\pi i M_j \alpha_j})^{-1} \quad (4.2)$$

with  $g_1 \in R(G)$ .

If  $p \in V_2$  then there exists an edge  $e$  issuing from  $p$  such that  $\alpha_e = m\alpha_1$  with  $m \in \mathbb{Z} - \{0\}$ ; let  $q = t(e)$ . Since  $\alpha_{\bar{e}} = -\alpha_e$  it follows that  $q \in V_2$  as well and thus the vertices in  $V_2$  can be paired as above.



Let  $e_k, k = 1, \dots, d$  and  $e'_k, k = 1, \dots, d$  be the edges issuing from  $p$  and  $q$  respectively, with  $e_d = e$ ,  $e' = \bar{e}$ . Then, by (1.4), the  $e_k$ 's can be ordered so that

$$r_e(e^{2\pi i \alpha_{e_k}}) = r_e(e^{2\pi i \alpha_{e'_k}}), \text{ for all } k = 1, \dots, d-1$$

which implies that

$$1 - e^{2\pi i \alpha_{e_k}} \equiv 1 - e^{2\pi i \alpha_{e'_k}} \pmod{1 - e^{2\pi i \alpha_e}}. \quad (4.3)$$

Similarly, from

$$r_e(f_p) = r_e(f_q)$$

we deduce that

$$f_q \equiv f_p \pmod{1 - e^{2\pi i \alpha_e}}. \quad (4.4)$$

The part of (1.10) corresponding to  $p$  and  $q$ ,

$$f_p \prod_{j=1}^d (1 - e^{2\pi i \alpha_{e_j}})^{-1} + f_q \prod_{j=1}^d (1 - e^{2\pi i \alpha_{e'_j}})^{-1},$$

can be expressed as

$$\frac{f_p \prod_{j=2}^N (1 - e^{2\pi i \alpha_{e'_j}}) - e^{2\pi i \alpha_1} f_q \prod_{j=2}^N (1 - e^{2\pi i \alpha_{e_j}})}{(1 - e^{2\pi i \alpha_1}) \prod_{j=2}^N (1 - e^{2\pi i \alpha_{e_j}}) \prod_{j=2}^N (1 - e^{2\pi i \alpha_{e'_j}})}. \quad (4.5)$$

From the congruences (4.3) and (4.4) we conclude that  $1 - e^{2\pi i \alpha_1}$  divides the numerator of (4.5), so we deduce that

$$f_p \prod_{j=1}^d (1 - e^{2\pi i \alpha_{e_j}})^{-1} + f_q \prod_{j=1}^d (1 - e^{2\pi i \alpha_{e'_j}})^{-1} = \frac{g_{p,q}}{\prod_{j=2}^N (1 - e^{2\pi i M_j \alpha_j})}$$

with  $g_{p,q} \in R(G)$ . Therefore

$$\sum_{p \in V_2} f(p) \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} = \frac{g_2}{\prod_{j=2}^N (1 - e^{2\pi i M_j \alpha_j})} \quad (4.6)$$

with  $g_2 \in R(G)$ . Adding (4.2) and (4.6) we obtain

$$\frac{g}{\prod_{j=1}^N (1 - e^{2\pi i M_j \alpha_j})} = \frac{g_1 + g_2}{\prod_{j=2}^N (1 - e^{2\pi i M_j \alpha_j})}$$

with  $g_1 + g_2 \in R(G)$ , hence  $1 - e^{2\pi i M_1 \alpha_1}$  divides  $g$ . The same argument can be used to show that each  $1 - e^{2\pi i M_j \alpha_j}$  divides  $g$ .

The proof of the theorem now follows from:

**Lemma 4.1.** *If  $P \in R(G)$  and  $\alpha, \beta$  are linearly independent weights such that  $1 - e^{2\pi i \alpha}$  divides  $(1 - e^{2\pi i \beta})P$ , then  $1 - e^{2\pi i \alpha}$  divides  $P$ .*

**Lemma 4.2.** *If  $P \in R(G)$  and  $\beta_1, \dots, \beta_k$  are pairwise linearly independent weights such that  $1 - e^{2\pi i \beta_j}$  divides  $P$  for all  $j = 1, \dots, k$  then*

$$(1 - e^{2\pi i \beta_1}) \dots (1 - e^{2\pi i \beta_k}) \text{ divides } P.$$

□

*The proof of Theorem 1.2. :*

Let  $\alpha$  be a weight that is not in the convex hull of  $\{\alpha_p; p \in V_\Gamma\}$ . Then there exists  $\xi \in \mathfrak{g}$  and  $p_0 \in V_\Gamma$  such that  $(\alpha - \alpha_{p_0})(\xi) < 0$  and  $(\alpha_p - \alpha_{p_0})(\xi) > 0$  for all  $p \neq p_0$ . If  $e \in E_\Gamma$  and  $\alpha_e(\xi) < 0$  then

$$(1 - e^{2\pi i \alpha_e})^{-1} = -e^{2\pi i \alpha_e} (1 - e^{2\pi i \alpha_e})^{-1},$$

and using this we deduce that

$$\chi(f) = \sum_{p \in V} (-1)^{\sigma_p} e^{2\pi i (\alpha_p - \sum' \alpha_e)} \prod_{e \in \mathcal{E}_p} (1 - e^{2\pi i \alpha_e})^{-1}, \quad (4.7)$$

where  $\sum' \alpha_e$  in the exponent is the sum

$$\sum_{\substack{i(e)=p \\ \alpha_e(\xi) < 0}} \alpha_e = - \sum_{\substack{t(e)=p \\ \alpha_e(\xi) > 0}} \alpha_e = \delta_p^\# - \delta_p. \quad (4.8)$$

From (4.7) and (4.8) we deduce that

$$\chi(f) = \sum_{p \in V} (-1)^p e^{2\pi i (\alpha_p - \sum' \alpha_e)} \prod_{e \in \mathcal{E}_p} \left( \sum_{k_e \geq 0} e^{2\pi i k_e \alpha_e} \right). \quad (4.9)$$

Suppose  $\alpha$  is a weight of  $Q(f)$ ; then there exists  $p \in V_\Gamma$  and non-negative integers  $\{k_e\}_{e \in \mathcal{E}_p}$  such that

$$\alpha = \alpha_p + \sum_{\substack{t(e)=p \\ \alpha_e(\xi) > 0}} \alpha_e + \sum_{e \in \mathcal{E}_p} k_e \alpha_e, \quad (4.10)$$

which implies that

$$\alpha - \alpha_{p_0} = \alpha_p - \alpha_{p_0} + \sum_{\substack{t(e)=p \\ \alpha_e(\xi) > 0}} \alpha_e + \sum_{e \in \mathcal{E}_p} k_e \alpha_e. \quad (4.11)$$

But when we evaluate (4.11) at  $\xi$ , the right hand side is non-negative, while the left hand side is strictly negative ! This contradiction proves that  $\alpha$  is not a weight of  $Q(f)$ .  $\square$

*The proof of Theorem 1.3. :*

Let  $\alpha = \alpha_{p_0}$  be an extremal weight, i.e. a vertex of  $\Delta$ . Then there exists  $\xi \in \mathfrak{g}$  such that  $(\alpha_p - \alpha_{p_0})(\xi) > 0$  for all  $p \neq p_0$ . In this case (4.11) implies

$$0 = (\alpha_p - \alpha_{p_0})(\xi) + \sum_{\substack{t(e)=p \\ \alpha_e(\xi) > 0}} \alpha_e(\xi) + \sum_{e \in \mathcal{E}_p} k_e \alpha_e(\xi). \quad (4.12)$$

Since each term on the right hand side is non-negative, (4.12) is only true if

1.  $p = p_0$  (which also implies that  $\alpha_e(\xi) < 0$  for all  $e$  with  $t(e) = p$ , i.e. that there are no terms in the first sum), and
2.  $k_e = 0$  for all  $e \in \mathcal{E}_p$ .

This proves that the multiplicity with which  $\alpha$  occurs in  $Q(f)$  is 1.  $\square$

*The proof of Theorem 1.4. :*

From (4.8) and (4.10), the multiplicity with which a weight  $\alpha$  appears in the term

corresponding to the vertex  $p$  is equal to  $(-1)^p$  times the number of distinct ways in which  $\alpha - \alpha_p + \delta_p^\# - \delta_p$  can be written as a sum

$$\sum_{e \in \mathcal{E}_p} k_e \alpha_e,$$

with  $k_e$ 's non-negative integers; and this number is  $N_p(\alpha - \alpha_p + \delta_p^\# - \delta_p)$ . Counting the contributions given by all the vertices we obtain (1.14).  $\square$

## 5. THE RESIDUE OPERATION

Let  $G$  be an  $n$ -dimensional torus, let  $T$  be a circle subgroup of  $G$ , and let

$$\chi_k = e^{2\pi i \alpha_k}, \quad k = 1, \dots, d$$

be characters of  $G$  and  $f$  an element of the character ring,  $R(G)$ . The goal of this section is to make sense of the integral

$$\int_T \frac{f(gt)}{\prod (1 - \chi_k(gt))} dt \quad (5.1)$$

as a function of  $g \in G$ . If the restriction of  $\chi_k$  to  $T$  is identically one, the denominator in the integrand is identically zero when  $\chi_k(g) = 1$ . Hence, for (5.1) to make sense, we are forced to assume that the restriction of  $\chi_k$  to  $T$  is *not* identically one. Even with this assumption, however, the integrand has poles at the points where  $\chi_k(gt) = 1$ ; so to make sense of (5.1) we must “regularize” this integral and this we will do as follows. Fix a basis vector,  $\xi$  of  $\mathbb{Z}_T$ , and identify  $T$  with  $S^1$  via the map

$$\exp(s\xi) \rightarrow e^{2\pi i s}.$$

Then, with  $z = e^{2\pi i x}$ , the integrand of (5.1) becomes a meromorphic function

$$f^\#(gz) = f(gz) \prod_{k=1}^d (1 - \chi_k(gz))^{-1} \quad (5.2)$$

on the complex plane with poles on the unit circle. Now move the contour of integration from the unit circle to a contour surrounding these poles, *e.g.* a circle of radius greater than one oriented in a *counter-clock-wise* sense plus a circle of radius less than one oriented in a *clock-wise* sense. In other words, replace (5.1) by the integral

$$\frac{1}{2\pi i} \int_{C_+} f^\#(gz) \frac{dz}{z} \quad (5.3)$$

minus the integral

$$\frac{1}{2\pi i} \int_{C_-} f^\#(gz) \frac{dz}{z}, \quad (5.4)$$

$C_+$  being a circle of radius greater than one and  $C_-$  a circle of radius less than one, both these circle being oriented in a counter-clock-wise sense. Let us denote this regularized integral, *i.e.* the difference of (5.3) and (5.4), by  $(Res_T f^\#)(g)$ . It is easy to see that this function is  $T$ -invariant,

$$(Res_T f^\#)(gt) = (Res_T f^\#)(g)$$

and hence defines a function on  $G/T$ . We will prove:

**Theorem 5.1.**  *$Res_T f^\#$  is an element of  $R(G/T)$ .*

**Remark.** The definition of  $Res_T f^\#$  depends on the identification of  $S^1$  with  $T$  given by  $\exp(x\xi) \leftrightarrow e^{2\pi i x}$ . If we replace  $\xi$  by  $-\xi$ , the orientations of the circles,  $C_+$  and  $C_-$ , will get reversed, and hence this will change the signs of (5.3) and (5.4).

In proving Theorem 5.1, we can assume without loss of generality that  $f = e^{2\pi i \alpha}$ ,  $\alpha \in \mathbb{Z}_G^*$ . Let  $e_1, \dots, e_n$  be a basis of  $\mathbb{Z}_G$  with  $\xi = e_n$  and let  $y_1, \dots, y_{n-1}$  and  $x$  be the coordinates on  $\mathfrak{g}$  associated with this basis. We can then write

$$\alpha_i = k_i x + \beta_i(y) \quad (5.5)$$

and

$$\alpha = kx + \beta(y) \quad (5.6)$$

with  $k_i = \alpha_i(\xi)$  and  $k = \alpha(\xi)$ . Thus letting

$$z = e^{2\pi i x} \quad (5.7)$$

$$a_i = e^{2\pi i \beta_i(y)} \quad (5.8)$$

$$b = e^{2\pi i \beta(y)} \quad (5.9)$$

the integrals (5.3) and (5.4) become

$$\frac{1}{2\pi i} \int_{C_+} \frac{bz^k}{\prod (1 - a_i z^{k_i})} \frac{dz}{z} \quad (5.10)$$

and

$$\frac{1}{2\pi i} \int_{C_-} \frac{bz^k}{\prod (1 - a_i z^{k_i})} \frac{dz}{z}. \quad (5.11)$$

Therefore, to prove the theorem, it suffices to show that each of these integrals individually is in  $R(G/T)$ . To verify this for (5.11), let's order the factors in the denominator of the integrand so that  $k_i = -k'_i < 0$  for  $1 \leq i \leq r$  and  $k_i > 0$  for  $r+1 \leq i \leq d$ . This integrand is then equal to

$$\frac{bz^{k'-1}}{\prod_{i=1}^r (z^{k'_i} - a_i) \prod_{i=r+1}^d (1 - a_i z^{k_i})} \quad (5.12)$$

with  $k' = k - k_1 - \dots - k_r$ . Hence, if  $k' > 0$ , (5.12) is *holomorphic* at zero; so, in particular:

**Lemma 5.1.** *The integral (5.4) is zero if  $k > k_1 + \dots + k_r$ .*

For  $1 \leq i \leq r$  and  $z \approx 0$ , let  $a'_i = a_i^{-1}$  and let

$$S_i(z) = \frac{1}{z^{k'_i} - a_i} = \frac{-a_i}{1 - a'_i z^{k'_i}} = -a_i \sum_{l=0}^{\infty} (a'_i z^{k'_i})^l, \quad (5.13)$$

and, for  $r+1 \leq i \leq d$ , let

$$S_i(z) = \frac{1}{1 - a_i z^{k_i}} = \sum_{l=0}^{\infty} (a_i z^{k_i})^l. \quad (5.14)$$

Then the integral (5.11) is just the degree -1 term in the Laurent series

$$bz^{k'-1} \prod_{i=1}^d S_i(z) \quad (5.15)$$

and this term is clearly a polynomial in  $b, a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}$ , and  $a_{r+1}, \dots, a_d$ , with integer coefficients. Hence, by (5.8) and (5.9), it is clearly a trigonometric polynomial in  $y_1, \dots, y_{n-1}$ . From this, together with Lemma 5.1, we conclude:

**Theorem 5.2.** *The integral (5.4) is an element of  $R(G/T)$ . Moreover, if  $f = e^{2\pi i \alpha}$  and  $k = \alpha(\xi)$ , this integral is zero if  $k > k_1 + \dots + k_r$ .*

To evaluate the integral (5.10) we make the substitution,  $z \rightarrow z^{-1}$  and reduce this integral to an integral of the type we've just evaluated. We conclude

**Theorem 5.3.** *The integral (5.3) is an element of  $R(G/T)$ . Moreover, if  $f = e^{2\pi i \alpha}$  and  $k = \alpha(\xi)$ , this integral is zero if  $k < k_{r+1} + \dots + k_d$ .*

Since  $k_i$  is negative for  $1 \leq i \leq r$  and positive for  $r+1 \leq i \leq d$  we have, in particular:

**Proposition 5.1.** *If  $f = e^{2\pi i \alpha}$  with  $k = \alpha(\xi)$ , the integral (5.4) is zero when  $k$  is positive and the integral (5.3) is zero when  $k$  is negative. Moreover, if  $k = 0$ , (5.4) is zero when  $r > 0$  and (5.3) is zero when  $d - r > 0$ .*

Suppose now that the weights,  $\alpha_i$ ,  $i = 1, \dots, d$  are *pairwise linearly independent*, i.e. suppose that  $\alpha_i$  and  $\alpha_j$  are linearly independent for  $i \neq j$ . Then the integrand in (5.10) - (5.11):

$$bz^{k-1} \prod_{i=1}^d (1 - a_i z^{k_i})^{-1} \quad (5.16)$$

has simple poles on the unit circle for generic values of  $y$ . (Recall that since  $a_k = e^{2\pi i \beta_k(y)}$ , the location of these poles depends on  $y$ .) Thus, one can compute the difference between (5.10) and (5.11) by computing the residues of (5.16) at these poles. We will show that the sum of these residues, which is, by definition, the regularized integral (5.1), is given by an expression involving the  $K$ -theoretic push-forward which we described in Section 3. More explicitly, let  $G_i$  be the kernel of the homomorphism  $\chi_i : G \rightarrow S^1$  and let  $r_i$  be the restriction map  $R(G) \rightarrow R(G_i)$  and  $\pi_i$  the projection of  $G_i$  onto  $G/T$ . We will prove:

**Theorem 5.4.** *Let  $f^\#$  be the function (5.2) and let*

$$\hat{f}_i = f \prod_{j \neq i} (1 - \chi_j)^{-1}.$$

*Then*

$$Res_T f^\# = \sum_{i=1}^r (\pi_i)_* r_i \hat{f}_i - \sum_{i=r+1}^d (\pi_i)_* r_i \hat{f}_i. \quad (5.17)$$

*Proof.* Let  $\theta_1, \dots, \theta_d$  be real numbers, let  $k_1, \dots, k_d$  be integers and let  $a_i = e^{2\pi i \theta_i}$ . As above we will order the  $k_i$ 's so that  $k_i < 0$  for  $1 \leq i \leq r$  and  $k_i > 0$  for  $r+1 \leq i \leq d$ . Let  $g(z)$  be the function (5.16), i.e.

$$g(z) = bz^{k-1} \prod_{i=1}^d (1 - a_i z^{k_i})^{-1}.$$

**Lemma 5.2.** *Suppose that, for  $i \neq j$ ,  $\theta_i, \theta_j$  and 1 are linearly independent over the rationals. Then  $g(z)$  has simple poles on the unit circle.*

*Proof.* Let

$$\omega_i = e^{2\pi i/k_i} \quad \text{and} \quad a_i^{-1/k_i} = e^{-2\pi i\theta_i/k_i}.$$

Then these poles are at the points

$$\omega_i^l a_i^{-1/k_i}, \quad 1 \leq l \leq k_i, \quad 1 \leq i \leq d; \quad (5.18)$$

so if  $\theta_i, \theta_j$  and 1 are linearly independent over the rationals these poles are distinct.  $\square$

Let us compute the residue of  $g(z)$  at the pole (5.18). The quotient

$$\frac{z - \omega_i^l a_i^{-1/k_i}}{1 - a_i z^{k_i}}$$

evaluated at  $z = \omega_i^l a_i^{-1/k_i}$  is equal, by l'Hopital's rule, to:

$$\frac{1}{-a_i k_i z^{k_i-1}} \quad \text{or, alternatively} \quad \frac{z}{-a_i k_i z^{k_i}}$$

evaluated at  $z = \omega_i^l a_i^{-1/k_i}$ , and since  $(\omega_i^l a_i^{-1/k_i})^{k_i} = a_i^{-1}$ , this quotient is just

$$-\frac{1}{k_i} \omega_i^l a_i^{-1/k_i}. \quad (5.19)$$

Thus the residue at  $z = \omega_i^l a_i^{-1/k_i}$  of the function

$$g(z) = \frac{1}{1 - a_i z^{k_i}} b z^{k-1} \prod_{j \neq i} (1 - a_j z^{k_j})^{-1}$$

is just

$$-\frac{b}{k_i} (\omega_i^l a_i^{-1/k_i})^k \prod_{j \neq i} (1 - a_j (a_i^{-1/k_i})^{k_j})^{-1}$$

which, if we set

$$b_i = b a_i^{-k/k_i} \quad (5.20)$$

and

$$a_{j,i} = a_j a_i^{-k_j/k_i}, \quad (5.21)$$

can be written

$$-\frac{1}{k_i} (\omega_i^l)^k b_i \prod_{j \neq i} (1 - (\omega_i^l)^{k_j} a_{j,i})^{-1}. \quad (5.22)$$

We will now show that if we give  $b$  and  $a_i$  the values (5.8) - (5.9) the sum of these residues is identical with the right hand side of (5.17). If  $b$  is equal to (5.9) and  $a_i$  is equal to (5.8), then by (5.5) and (5.6)

$$b_i = e^{2\pi i \sigma_i} \quad (5.23)$$

and

$$a_{j,i} = e^{2\pi i \alpha_{j,i}}, \quad (5.24)$$

where

$$\sigma_i = \alpha - \frac{k}{k_i} \alpha_i \quad (5.25)$$

and

$$\alpha_{j,i} = \alpha_j - \frac{k_j}{k_i} \alpha_i. \quad (5.26)$$

Let's now give a more "intrinsic" definition of  $\sigma_i$  and  $\alpha_{j,i}$ : Let  $\mathfrak{g}_i$  be the Lie algebra of the group,  $G_i$ , and  $\mathfrak{t}$  the Lie algebra of  $T$ . Since  $G_i$  is by definition the kernel of the homomorphism,  $e^{2\pi i \alpha_i} : G \rightarrow S^1$ ,  $\mathfrak{g}_i$  is the annihilator of  $\alpha_i$ ; so, by (5.25),  $\sigma_i$  is the *unique* element of  $\mathfrak{g}^*$  which is annihilated by  $\mathfrak{t}$  and has the same restriction to  $\mathfrak{g}_i$  as  $\alpha$ . Similarly,  $\alpha_{j,i}$  is the unique element of  $\mathfrak{g}^*$  which is annihilated by  $\mathfrak{t}$  and has the same restriction to  $\mathfrak{g}_i$  as  $\alpha_j$ . Note, by the way, that since  $\sigma_i$  and  $\alpha_{j,i}$  are annihilated by  $\mathfrak{t}$ , they are in the dual vector space to  $\mathfrak{g}/\mathfrak{t}$ ; or, in other words, in the dual of the Lie algebra of  $G/T$ .

Consider the kernel of the map  $G_i \rightarrow G/T$ . This consists of the elements

$$\exp\left(\frac{l}{k_i} \xi\right), \quad l = 1, \dots, k_i$$

and by (5.5) and (5.6)

$$e^{2\pi i \alpha}(\exp(\frac{l}{k_i} \xi)) = (\omega_i^l)^k \quad (5.27)$$

and

$$e^{2\pi i \alpha_j}(\exp(\frac{l}{k_i} \xi)) = (\omega_i^l)^{k_j}. \quad (5.28)$$

Thus the sum

$$-\frac{1}{k_i} \sum_{l=1}^{k_i} (\omega_i^l)^k e^{2\pi i \sigma_i} \prod_{j \neq i} (1 - (\omega_i^l)^{k_j} e^{2\pi i \alpha_{j,i}})^{-1}$$

of the residues of  $g(z)$  over the poles  $(\omega_i^l)^k a_i^{-1/k_i}$ ,  $1 \leq l \leq k_i$  is by formula (3.9) identical to the expression

$$-(\pi_i)_* r_i \frac{e^{2\pi i \alpha}}{\prod_{j \neq i} (1 - e^{2\pi i \alpha_j})}$$

if  $r+1 \leq i \leq d$  (in which case  $k_i = |k_i|$ ) and is equal to

$$(\pi_i)_* r_i \frac{e^{2\pi i \alpha}}{\prod_{j \neq i} (1 - e^{2\pi i \alpha_j})}$$

when  $1 \leq i \leq r$  (in which case  $k_i = -|k_i|$ ).  $\square$

## 6. QUANTIZATION COMMUTES WITH REDUCTION

We will prove below Theorems 1.7 and 1.8 of Section 1. As in Theorem 1.7, let  $f$  be an element of  $K_G(\Gamma)$ , let  $\phi : V \rightarrow \mathbb{R}$  be a  $T$ -moment map, let  $c$  be a regular value of  $\phi$  and let  $e$  be an oriented edge of  $\Gamma$  with  $\phi(q) > c > \phi(p)$ , where  $p = i(e)$  and  $q = t(e)$  (*i.e.*  $e$  corresponds to a vertex of the hypergraph,  $\Gamma_c$ ; we will denote this vertex by  $e$ , as well.)

Consider the expressions

$$\hat{f}_e = f_p \prod_{e'} (1 - e^{2\pi i \alpha_{e'}})^{-1} \quad (6.1)$$

$$\hat{f}_{\bar{e}} = f_q \prod_{e''} (1 - e^{2\pi i \alpha_{e''}})^{-1} \quad (6.2)$$

the product in (6.1) being over all edges,  $e' \neq e$ , with  $i(e') = p$ , and the product in (6.2) being over all edges,  $e'' \neq \bar{e}$ , with  $i(e'') = q$ .

**Lemma 6.1.** *Let  $r_e = r_{\bar{e}}$  be the restriction map  $R(G) \rightarrow R(G_e)$ . Then*

$$r_e \hat{f}_e = r_{\bar{e}} \hat{f}_{\bar{e}}. \quad (6.3)$$

Let  $\pi_e = \pi_{\bar{e}}$  be the projection of  $G_e$  onto  $G/T$ . As a corollary of Lemma 6.1 we get two alternative ways of defining (1.18):

$$(\pi_e)_* r_e \hat{f}_e = (\pi_{\bar{e}})_* r_{\bar{e}} \hat{f}_{\bar{e}} = f_c^\#(e), \quad (6.4)$$

and, as a consequence of (6.4), the following theorem:

**Theorem 6.1.** *Let  $c$  and  $c'$  be regular values of  $\phi$ . Suppose there exists just one vertex,  $p$ , with  $c < \phi(p) < c'$ . Then*

$$\chi_c(f) - \chi_{c'}(f) = \text{Res}_T \left( f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} \right) \quad (6.5)$$

*Proof.* If  $e \in V_c$  and  $t(e) \neq p$ , then  $e \in V_{c'}$ , and if  $e \in V_{c'}$  and  $i(e) \neq p$ , then  $e \in V_c$ . Moreover, in both cases,

$$f_c^\#(e) = f_{c'}^\#(e), \quad (6.6)$$

by (6.4). Thus, if  $e_i$ ,  $i = 1, \dots, r$  are the elements of  $V_c$  with  $t(e_i) = p$ , and  $e_i$ ,  $i = r+1, \dots, d$ , are the elements of  $V_{c'}$  with  $i(e_i) = p$ , the difference between  $\chi_c(f)$  and  $\chi_{c'}(f)$  is, by (6.4), equal to

$$\sum_{i=1}^r f_c^\#(e_i) - \sum_{i=r+1}^d f_{c'}^\#(e_i),$$

or, also by (6.4), to

$$\sum_{i=1}^r (\pi_{e_i})_* r_{e_i} \hat{f}_{e_i} - \sum_{i=r+1}^d (\pi_{e_i})_* r_{e_i} \hat{f}_{e_i},$$

which, by (5.17), is identical with

$$\text{Res}_T \left( f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} \right). \quad \square$$

To prove Theorem 1.7, let  $c_0 < c_1 < \dots < c_N$  be regular values of  $\phi$  with  $c_0 = c$ ,  $c_N$  greater than  $\phi_{max}$ , and with only one critical point,  $p_i$ , between  $c_i$  and  $c_{i+1}$ . Then

$$\chi_c(f) = \sum_{i=0}^N (\chi_{c_i}(f) - \chi_{c_{i+1}}(f)) = \sum_{\phi(p) > c} \text{Res}_T \left( f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} \right),$$

proving Theorem 1.7.

To prove Theorem 1.8, let  $f$  be an element of  $K_G(\Gamma)$  of the form (1.12) - (1.13) and let  $\phi : V_\Gamma \rightarrow \mathbb{R}$ ,  $\phi(p) = \alpha_p(\xi)$ . Then

$$\chi(f) = \sum_p e^{2\pi i \alpha_p} \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1}.$$



Identify  $T^{\mathbb{C}}$  with  $\mathbb{C} - 0$  and let  $C$  be a circle in the complex plane with radius greater than one oriented in a counter-clock-wise sense. Then

$$\frac{1}{2\pi i} \int_C \chi(f)(gz) \frac{dz}{z} = \sum_p \frac{1}{2\pi i} \int_C \left( \frac{e^{2\pi i \alpha_p}}{\prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})} \right) (gz) \frac{dz}{z}. \quad (6.7)$$

The right hand side of this identity is easy to evaluate: By Proposition 5.1, the summands with  $\alpha_p(\xi) < 0$  are zero, and the summands with  $\alpha_p(\xi) > 0$  are equal to

$$\text{Res}_T \left( f_p \prod_{i(e)=p} (1 - e^{2\pi i \alpha_e})^{-1} \right),$$

so by Theorem 1.7 the right hand side is equal to  $\chi_{red}(f)$ . As for the left hand side, by Theorem 1.1,  $\chi(f)$  is in  $R(G)$ ; so it is a finite sum of the form

$$\sum m_k e^{2\pi i \alpha_k}$$

with  $m_k \in \mathbb{Z}$  and  $\alpha_k \in \mathbb{Z}_G^*$ ; and the integral over  $C$  of the  $k$ -th term is zero except when  $e^{2\pi i \alpha_k}$  doesn't depend on  $z$ , in which case the integral is just  $2\pi i e^{2\pi i \alpha_k}$ . Hence the left hand side is equal to

$$\sum_{\alpha_k(\xi)=0} m_k e^{2\pi i \alpha_k},$$

which is the character of the representation,  $Q(\Gamma)^T$ .

## 7. GKM MANIFOLDS

Let  $(M, \omega)$  be a compact  $2d$ -dimensional symplectic manifold and  $\tau : G \times M \rightarrow M$  a Hamiltonian action of  $G$  on  $M$ . We will say that  $M$  is a *symplectic GKM manifold* if  $M^G$  is finite and if, for every  $p \in M^G$ , the weights  $\alpha_{i,p} \in \mathbb{Z}_G^*$ ,  $i = 1, \dots, d$  of the isotropy representation of  $G$  on  $T_p M$  are pair-wise linearly independent. Let

$$M^{(1)} = \{p \in M; \dim G_p \geq n - 1\}.$$

This set is called the *one-skeleton* of  $M$ ; and  $M$  is a GKM manifold if and only if  $M^{(1)}$  consists of  $G$ -invariant imbedded 2-spheres, each of which contains exactly two fixed points. These 2-spheres can intersect at the fixed points; so the combinatorial structure of  $M^{(1)}$  is that of a graph,  $\Gamma$ , having the fixed points of  $\tau$  as vertices and these 2-spheres as edges. For each oriented edge,  $e$ , of  $\Gamma$ , let  $\varrho_e$  be the isotropy representation of  $G$  on the tangent space to this 2-sphere at the fixed point,  $t(e)$ ; and for each vertex,  $p$ , of  $\Gamma$  let  $\tau_p$  be the isotropy representation of  $G$  on  $T_p M$ . It is easily checked that  $\varrho$  and  $\tau$  have properties (1.2) - (1.4) and hence define an action of  $G$  on  $\Gamma$ .

For GKM manifolds the cohomology groups,  $H_G(\Gamma)$  and  $K_G(\Gamma)$ , turn out to be equal to cohomology groups of  $M$ . More explicitly, let  $H_G(M)$  be the equivariant cohomology ring of  $M$  with complex coefficients and let  $K_G(M)$  be the  $K$ -cohomology ring of  $M$ . Then there are ring homomorphisms

$$H_G(M) \simeq H_G(\Gamma), \quad (\text{see [GKM]}) \quad (7.1)$$

$$K_G(M) \otimes \mathbb{C} \simeq K_G(\Gamma) \otimes \mathbb{C}, \quad (\text{see [KR]}). \quad (7.2)$$

With (7.1) and (7.2) as our point of departure, we will briefly describe some geometric implications of the theorems proved in this paper. The first of our results, Theorem 1.1, is a "combinatorial" explanation of why the right hand side of the Atiyah-Bott fixed point formula makes sense, *i.e.* why (1.10) *does* define a character

of a virtual representation of  $G$ . Theorems 1.2 - 1.4 are, in the manifold setting, well-known results about the “quantum” action of  $G$  on  $M$ : Suppose  $[\omega] \in H^2(M, \mathbb{Z})$ . Then there exists a line bundle,  $\mathbb{L} \rightarrow M$ , and a connection,  $\nabla$ , on this bundle with  $\text{curv}(\nabla) = \omega$ ; and one says that the action,  $\tau$ , of  $G$  on  $M$  is *pre-quantizable* if it lifts to an action of  $G$  on  $\mathbb{L}$  preserving  $\nabla$ . Now equip  $M$  with a  $G$ -invariant Riemannian metric and let

$$\not{D}_{\mathbb{C}} : S_{\mathbb{C}}^+ \rightarrow S_{\mathbb{C}}^-$$

be the  $\text{spin}^{\mathbb{C}}$  Dirac operator. Given the connection,  $\nabla$ , one can twist this operator with operator with  $\mathbb{L}$  to get a Dirac operator

$$\not{D}_{\mathbb{C}}^{\mathbb{L}} : S_{\mathbb{C}}^+ \otimes \mathbb{L} \rightarrow S_{\mathbb{C}}^- \otimes \mathbb{L},$$

and the virtual vector space

$$Q(M) = \text{kernel}(\not{D}_{\mathbb{C}}^{\mathbb{L}}) - \text{cokernel}(\not{D}_{\mathbb{C}}^{\mathbb{L}}) \quad (7.3)$$

is called the *spin<sup>C</sup>-quantization* of  $M$ . From the action of  $G$  on  $\mathbb{L}$ , one gets a representation,  $\tau_Q$ , of  $G$  on this space, and its character,  $\text{trace}\tau_Q$ , is equal, by the Atiyah-Bott formula, to the formal character,  $\chi(f)$ , defined by (1.10),  $f$  being the element of  $K_G(\Gamma)$  corresponding to  $[\mathbb{L}]$  under the isomorphism (7.2).

For  $\tau_Q$ , the convexity theorem (Theorem 1.2) is due to Guillemin and Sternberg, who pointed out in [GS] that it can be deduced from “quantization commutes with reduction” and the Atiyah-Guillemin-Sternberg convexity theorem for moment maps. (However, the simple proof of this theorem described in Section 4 seems to have eluded them.) As for Theorem 1.4, for co-adjoint orbits this is the celebrated Kostant Multiplicity Theorem. Our proof of it in Section 4 is modeled on Cartier’s proof of Kostant’s theorem in [Ca] and the symplectic version of the proof described in [GLS].

Let  $T$  be a circle subgroup of  $G$  and let  $M_c$  be the reduction of  $M$  with respect to  $T$ . For  $f = [\mathbb{L}]$ , the “reduced” character,  $\chi_c(f)$ , in Theorem 1.5 can be shown, by the orbifold version of Atiyah-Bott, to be equal to the character of the representation of  $G/T$  on  $Q(M_c)$ . Our residue formula for it, (formula (1.20)) appears to be a new result even in the manifold case; however, the formula (1.21), which is a special case of this formula, is just the “quantization commutes with reduction” theorem for circle actions. A good reference for the long and entangled history of “ $[Q, R] = 0$ ” is the survey article [Sj]. For circle actions there are several relatively simple proofs, among them that of Duistermaat-Guillemin-Meinrenken-Wu ([DGMW]), Ginzburg-Guillemin-Karshon ([GGK]) and Metzler ([Me]). Of these, Metzler’s proof is probably the closest in spirit to our combinatorial proof of Theorem 1.8 in Section 6.

## REFERENCES

- [AB] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes, I*, Ann. of Math. **86** (1967), 374-407.
- [Ca] P. Cartier, *On H. Weyl’s character formula*, Bull. Amer. Math. Soc. **67** (1961), 228-230.
- [DGMW] H. Duistermaat, V. Guillemin, E. Meinrenken and S. Wu, *Symplectic reduction and Riemann-Roch for circle actions*, Math. Res. Lett. **2** (1995), no. 3, 259-266.
- [GGK] V. Ginzburg, V. Guillemin and Y. Karshon, *Cobordism theory and localization formulas for Hamiltonian group actions*, Internat. Math. Res. Notices 1996, no. 5, 221-234.
- [GKM] M. Goresky, R. Kottwitz and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25-83.

- [GLS] V. Guillemin, E. Lerman and S. Sternberg, *On the Kostant multiplicity formula*, J. Geom. Phys. **5** (1988), no. 4, 721-750.
- [GS] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491-513.
- [GZ1] V. Guillemin and C. Zara, *Equivariant de Rham theory and graphs*, Asian J. of Math. **3** (1999), no. 1, 49-76.
- [GZ2] V. Guillemin and C. Zara, *One-skeleta, Betti numbers and Equivariant Cohomology*, math.DG/9903051, to appear in Duke Math. J.
- [GZ3] V. Guillemin and C. Zara, *Morse Theory on Graphs*, math.CO/0007161
- [KR] A. Knutson and I. Rosu, *Equivariant K-theory from Equivariant Cohomology and GKM manifolds*, math.AT/9912088.
- [Me] D. Metzler, *A K-theoretic note on geometric quantization*, Manuscripta Math. **100** (1999), no. 3, 277-289.
- [Sj] R. Sjamaar, *Symplectic reduction and Riemann-Roch formulas for multiplicities* Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 3, 327-338.

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